Some remarks on the characterization of Fibonacci and Lucas numbers

Summary: We introduce a smart representation of Fibonacci and Lucas numbers and show how formulas about these sequences can be derived systematically. As an application we prove a characterization of Fibonacci and Lucas numbers by the roots of a 2-dimensional 4-th order polynomial. Further we establish some generalizations of the Millin series.

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1. Introduction

For easy reference we first list the definitions used throughout this work.

Definition 1-1

Fibonacci numbers are denoted by f_n.

Lucas numbers are denoted by l_n.

According to the same index n we say l_n *is corresponding to* f_n *(and vice versa).*

Definition 1-2

The golden ratio $\frac{1}{2}\left(1+\sqrt{5}\right)$ *will be referenced by* ϕ.

The natural logarithm of the golden ratio will be denoted by $\psi = \ln\phi$.

Looking to Binet's formula for Fibonacci numbers, we have

$$\begin{aligned} f_n &= \frac{\phi^n - (-\phi)^{-n}}{\phi - (-\phi)^{-1}} \\ &= \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \end{aligned}.$$

By definition of sine and cosine hyberbolic, it follows therefore

(1-1) $$f_n = \begin{cases} \frac{2}{\sqrt{5}}\sinh(n\psi), & \text{if } n \text{ is even} \\ \frac{2}{\sqrt{5}}\cosh(n\psi), & \text{if } n \text{ is odd} \end{cases}$$

For the Lucas numbers we can easily deduce a very similar formula:

(1-2) $$l_n = \begin{cases} 2\cosh(n\psi), & \text{if } n \text{ is even} \\ 2\sinh(n\psi), & \text{if } n \text{ is odd} \end{cases}$$

Regarding these relations, many formulas for Fibonacci and Lucas numbers easily follow from the rich treasury of appropriate sinh and cosh formulas.

For example, from the basic identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

we can derive

$$\left(\frac{l_n}{2}\right)^2 - \left(\frac{\sqrt{5}}{2} f_n\right)^2 = (-1)^n$$

by setting the representations above and regarding the cases with odd and with even *n*. From this we get the well known fundamental identity

$$l_n^{\ 2} - 5 f_n^{\ 2} = 4 \cdot (-1)^n$$

without any further calculations.

Another example: the Moivre Theorem

$$(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$$

results in a multiple angle formula

$$\left(\left(\frac{l_m}{2}\right) + \left(\frac{\sqrt{5}}{2} f_m\right)\right)^n = \left(\frac{l_{mn}}{2}\right) + \left(\frac{\sqrt{5}}{2} f_{mn}\right).$$

Especially for n=2 we obtain

$$(l_m)^2 + 2\sqrt{5}\, l_m f_m + 5(f_m)^2 = 2 l_{2m} + 2\sqrt{5}\ f_{2m}$$

from which follows both the identities

$$l_m^2 + 5 f_m^2 = 2 l_{2m}$$

and

$$l_m f_m = f_{2m}\,.$$

In general, by binomial expansion we get

$$\begin{aligned}\left(l_m + \sqrt{5} f_m\right)^n &= 2^n \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{5} f_m\right)^k (l_m)^{n-k} \\ &= 2^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} 5^k (f_m)^{2k} (l_m)^{n-2k} + \sqrt{5} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} 5^k (f_m)^{2k+1} (l_m)^{n-2k-1}\end{aligned}.$$

Hence we obtain

$$l_{mn} = 2^{n-1} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} 5^k (f_m)^{2k} (l_m)^{n-2k}\ .$$

$$f_{mn} = 2^{n-1} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} 5^k (f_m)^{2k+1} (l_m)^{n-2k-1}\ .$$

2. Characterization of Fibonacci and Lucas numbers by a 4-th order polynomial

In this section we first characterize Fibonacci and Lucas numbers by square numbers. Based on this, we finally show that these numbers are the roots of a definite Diophantine polynomial. For the proof we make use of the representation introduced in section 1.

Theorem 2-1

Let P be a non-negative integer number. Then the following statements holds true

(i) *P is a Fibonacci number and there exists an even index n satisfying* $P = f_n$ *if and only if the term* $5P^2 + 4$ *is a square number.*

(ii) *P is a Fibonacci number and there exists an odd index n satisfying* $P = f_n$ *if and only if the term* $5P^2 - 4$ *is a square number.*

Proof: Let $P = f_n$ be a Fibonacci number with an even index *n*. Then $P = \frac{2}{\sqrt{5}}\sinh(n\psi)$ and it follows

$$5P^2 + 4 = 5\left(\frac{2}{\sqrt{5}}\sinh(n\psi)\right)^2 + 4 = 4\left(\sinh^2(n\psi) + 1\right) = \left(2\cosh(n\psi)\right)^2$$

where the latter is the square of the *n*-th Lucas number. This is (i) "⇒".

We come now to the opposite direction of (i). For *P*=0 the statement is trivially true, so we can restrict ourselves to *P*>0. Then, with

$$y := \operatorname{ar\,sinh}\left(\frac{\sqrt{5}}{2}P\right)$$

and

$$\nu := \frac{y}{\psi}$$

we obtain

$$P = \frac{2}{\sqrt{5}}\sinh(\nu\psi).$$

By definition y and ν both are positive. We are ready if we can show that ν is an integer and is even. In doing so, let *n* be the greatest even integer less than or equal to ν. Then

$$f_n := \frac{2}{\sqrt{5}}\sinh(n\psi).$$

is a Fibonacci number. It follows

(2-1)
$$\begin{aligned}\frac{2}{\sqrt{5}}\sinh((\nu - n)\psi) &= \frac{2}{\sqrt{5}}\sinh(\nu\psi)\cosh(n\psi) - \frac{2}{\sqrt{5}}\sinh(n\psi)\cosh(\nu\psi) \\ &= P\cdot\frac{1}{2}\sqrt{5f_n^2 + 4} - f_n\cdot\frac{1}{2}\sqrt{5P^2 + 4}\end{aligned}$$

By choice of *n* it is $0 \le \nu - n < 2$ which results in

$$0 \le \frac{2}{\sqrt{5}}\sinh((\nu - n)\psi) < \frac{2}{\sqrt{5}}\sinh(2\psi) = \frac{\phi^2 - \phi^{-2}}{\sqrt{5}} = 1.$$

We realize that the right hand side of (2-1) has integer value because m and $\sqrt{5m^2+4}$ are either even or odd simultaneously for all m in discussion (where $m = P$ or $m = f_n$). So we can conclude

$$\sinh((\nu - n)\psi) = 0$$

From which follows v=n immediately. Therefore we have proved that, P is a Fibonacci number with the desired property according to statement (i).

For the proof of (ii) we argue very similar. Let $P = f_n$ be a Fibonacci number with an odd index n. Then $P = \frac{2}{\sqrt{5}}\cosh(n\psi)$ and it follows

$$5P^2 - 4 = 5\left(\frac{2}{\sqrt{5}}\cosh(n\psi)\right)^2 - 4 = 4\left(\cosh^2(n\psi) - 1\right) = (2\sinh(n\psi))^2$$

where the latter is the square of the n-th Lucas number. This is (ii) “⇒”.

Now we treat the opposite direction of (ii). For P=1 the statement is trivially true, so we can restrict ourselves to P>1. Then, with

$$y := \operatorname{ar}\cosh\left(\frac{\sqrt{5}}{2}P\right)$$

and

$$\nu := \frac{y}{\psi}$$

we get

$$P = \frac{2}{\sqrt{5}}\cosh(\nu\psi).$$

By definition y and ν both are positive. We are ready, if we can show, that ν is an integer and is odd. In doing so, let n be the greatest odd integer less than or equal to ν. Then

$$f_n := \frac{2}{\sqrt{5}}\cosh(n\psi).$$

is a Fibonacci number. It follows

(2-2)
$$\begin{aligned}\frac{2}{\sqrt{5}}\sinh((\nu - n)\psi) &= \sinh(\nu\psi)\frac{2}{\sqrt{5}}\cosh(n\psi) - \sinh(n\psi)\frac{2}{\sqrt{5}}\cosh(\nu\psi)\\ &= \frac{1}{2}\sqrt{5P^2-4}\cdot f_n - \frac{1}{2}\sqrt{5f_n^2-4}\cdot P\end{aligned}$$

By choice of n it is $0 \le \nu - n < 2$ which leads us to

$$0 \le \frac{2}{\sqrt{5}}\sinh((\nu - n)\psi) < \frac{2}{\sqrt{5}}\sinh(2\psi) = \frac{\phi^2 - \phi^{-2}}{\sqrt{5}} = 1.$$

The right hand side of (2-2) has an integer value, because m and $\sqrt{5m^2-4}$ are either even or odd simultaneously for all m (where $m = P$ or $m = f_n$) in discussion. So we can conclude

$$\sinh((\nu - n)\psi) = 0$$

which implies *v*=*n*. Therefore, we have proved that, *P* is a Fibonacci number with the desired property according to statement (ii). ☐

Corollary 2-1

A non-negative integer P is a Fibonacci number if and only if $5P^2+4$ *or* $5P^2-4$ *is a square number.*

Theorem 2-2

Let P be a non-negative integer number. Then the following statements holds true

(i) *Q is a Lucas number and there exists an even index n satisfying* $Q=l_n$ *if and only if the term* $\frac{1}{5}\left(Q^2-4\right)$ *is a square number.*

(ii) *Q is a Lucas number and there exists an odd index n satisfying* $Q=l_n$ *if and only if the term* $\frac{1}{5}\left(Q^2+4\right)$ *is a square number.*

<u>Proof:</u> Let $Q=l_n$ be a Lucas number with an even index *n*. Then $P=2\cosh(n\psi)$ and if follows

$$\frac{Q^2-4}{5}=\frac{1}{5}\left((2\cosh(n\psi))^2-4\right)=\frac{4}{5}\left(\cosh^2(n\psi)-1\right)=\left(\frac{2}{\sqrt{5}}\sinh(n\psi)\right)^2$$

where the latter is the square of the *n*-th Fibonacci number. This is (i) "⇒".

Of course, the opposite direction of (i) may be proved directly very similar to the proof of Theorem 2-1 (i). For a shorter argumentation we make use of that Theorem and set

$$P:=\sqrt{\frac{1}{5}\left(Q^2-4\right)}$$

Then, the term $5P^2+4$ is a square number, and so, by Theorem 2-1, *P* is equal to a Fibonacci number f_n with an even index *n*. Thus we can conclude

$$Q=\sqrt{5P^2+4}=\sqrt{5\left(\frac{2}{\sqrt{5}}\sinh(n\psi)\right)^2+4}=2\sqrt{\sinh^2(n\psi)+1}=2\cosh(n\psi)$$

what shows, that *Q* is the *n*-th Lucas number.

Assertion (ii) may be proved using a very similar argumentation. ☐

Corollary 2-2

A non-negative integer Q is a Lucas number if and only if $\frac{1}{5}\left(Q^2+4\right)$ *or* $\frac{1}{5}\left(Q^2-4\right)$ *is a square number.*

Theorem 2-3

We define the following polynomial:

(2-3) $$F(x,y):=25x^4-10x^2y^2+y^4-16$$

For each pair of non-negative integer numbers (x_0,y_0) *the following statements are equivalent*

(i) (x_0,y_0) *is a root of F (i.e.* $F(x_0,y_0)=0$*).*

(ii) x_0 *is a Fibonacci number and* y_0 *is the corresponding Lucas number.*

Proof: As can be easily verified, we have

(2-4)
$$F(x,y) = \left(y^2 - 5x^2\right)^2 - 16 \\ = \left(\left(y^2 - 5x^2\right) - 4\right)\cdot\left(\left(y^2 - 5x^2\right) + 4\right)$$

Let (x_0, y_0) be a root of *F* with non-negative integer numbers x_0 and y_0, then by (2-4) we get

$$5x_0^2 + 4 = y_0^2 \quad \textit{or} \quad y_0^2 - 4 = 5x_0^2 \quad \text{respectively}$$

or

$$5x_0^2 - 4 = y_0^2 \quad \textit{or} \quad y_0^2 + 4 = 5x_0^2 \quad \text{respectively}$$

Obviously, by Corollary 2-1 and Corollary 2-2 this means, x_0 is a Fibonacci number and y_0 is a Lucas number. Thus there exists an index *n* satisfying $f_n = x_0$. Because of the fundamental identity $5f_n^2 + 4 = l_n^2$ it follows immediately $l_n = y_0$. Hence x_0 and y_0 are proved to be corresponding Fibonacci and Lucas numbers.

The opposite direction of the theorem plainly follows from the representation (2-4) of *F*. ☐

After Theorem 2-3 the non-negative integer roots of *F* plainly characterizes Fibonacci and Lucas numbers (more exact: *pairs of corresponding* Fibonacci and Lucas numbers). This means both, first, the *x*-part of each such root is a Fibonacci number whereas the *y*-part is a Lucas number, and, second, each pair (x_0, y_0) of corresponding Fibonacci and Lucas numbers is a root of *F*.

3. Generalizations of the Millin series

In this section we consider some generalizations of the Millin series in terms of the representation introduced in section 1. The Millin series $\sum_{n=0}^{\infty} \frac{1}{f_{2^n}}$ has sum $\frac{1}{2}\left(7 - \sqrt{5}\right)$. We extent the indices allowed to integer multiples of 2^n and will therefore prove the following result.

Theorem 3-1

The generalized Millin series has sum

(3-1)
$$\sum_{n=0}^{\infty} \frac{1}{f_{p2^n}} = \begin{cases} \frac{1}{2}\left(\frac{l_p + 2}{f_p} - \sqrt{5}\right), & \text{if } p \text{ even} \\ \frac{1}{2}\left(\frac{\left(l_p + 1\right)^2 + 3}{f_{2p}} - \sqrt{5}\right), & \text{if } p \text{ odd} \end{cases}$$

In terms of the golden ratio the sum (3-1) can also be expressed by

(3-2)
$$\sum_{n=0}^{\infty} \frac{1}{f_{p2^n}} = \begin{cases} \frac{1}{f_p} + \frac{1}{f_p}\phi^{-p}, & \text{if } p \text{ even} \\ \frac{1}{f_p} + \frac{\sqrt{5}}{l_p}\phi^{-p}, & \text{if } p \text{ odd} \end{cases}$$

Another compact form which is valid for odd p as well as for even p is given by

(3-3) $$\sum_{n=0}^{\infty}\frac{1}{f_{p2^n}}=\frac{1}{f_p}+\frac{\sqrt{5}}{\phi^{2p}-1}$$

For some special parameters we get simpler formulas. If we set $p=2q$ with an odd parameter q we find

(3-4) $$\sum_{n=0}^{\infty}\frac{1}{f_{q2^{n+1}}}=\frac{\sqrt{5}\varphi^{-q}}{l_q}, \quad q \text{ odd}$$

Similar, for $p=4q$ (q odd or even) we have

(3-5) $$\sum_{n=0}^{\infty}\frac{1}{f_{q2^{n+2}}}=\frac{\varphi^{-2q}}{f_{2q}}$$

Both relations ((3-4) and (3-5)) can be easily deduced from (3-3) using Binet's formula.

To come to the proof of Theorem 3-1 we first consider the following lemma, which will be proved in terms of hyberbolic functions. Especially, we are using a well known half angle formula for the cotangens hyperbolic.

Lemma 3-1

(i) $$\sum_{k=1}^{n}\frac{1}{\sinh(2^k x)}=\frac{e^{-x}}{\sinh(x)}-\frac{e^{-2^n x}}{\sinh(2^n x)}, \text{ for } x\in\Re,\ x\neq 0.$$

(ii) $$\sum_{k=1}^{\infty}\frac{1}{\sinh(2^k x)}=\frac{e^{-x}}{\sinh(x)}, \text{ for } x\in\Re,\ x\neq 0.$$

<u>Proof:</u> We have

$$\coth(x)=\frac{\cosh(2x)+1}{\sinh(2x)}=\coth(2x)+\frac{1}{\sinh(2x)}$$

which implies

(3-6) $$\frac{\cosh(2x)}{\sinh(2x)}=\frac{\cosh(x)}{\sinh(x)}-\frac{1}{\sinh(2x)}$$

and so, subtracting 1 on both sides, we get

(3-7) $$\frac{\cosh(2x)-\sinh(2x)}{\sinh(2x)}=\frac{\cosh(x)-\sinh(x)}{\sinh(x)}-\frac{1}{\sinh(2x)}$$

and finally

(3-8) $$\frac{1}{\sinh(2x)}=\frac{e^{-x}}{\sinh(x)}-\frac{e^{-2x}}{\sinh(2x)}$$

Thus we have

(3-9)
$$\sum_{k=1}^{n}\frac{1}{\sinh(2^k x)}=\sum_{k=1}^{n}\left(\frac{e^{-2^{k-1}x}}{\sinh(2^{k-1}x)}-\frac{e^{-2^k x}}{\sinh(2^k x)}\right)=\frac{e^{-x}}{\sinh(x)}-\frac{e^{-2^n x}}{\sinh(2^n x)}$$

which proves (i).

Formula (ii) easily follows from (i) because the term $\frac{e^{-2^n x}}{\sinh(2^n x)}$ tends to zero for $n\to\infty$. □

<u>Now we are able to prove Theorem 3-1:</u> With respect to formula (1-1) we have for n>0

$$f_{p2^n}=\frac{2}{\sqrt{5}}\sinh(p\cdot 2^n\psi)\text{, for n>0.}$$

Thus we get

(3-10)
$$\sum_{n=1}^{\infty}\frac{1}{f_{p2^n}}=\frac{\sqrt{5}}{2}\sum_{n=1}^{\infty}\frac{1}{\sinh(p\cdot 2^n\psi)}=\frac{\sqrt{5}}{2}\frac{e^{-p\psi}}{\sinh(p\psi)}=\frac{\sqrt{5}}{2}\frac{\phi^{-p}}{\sinh(p\psi)}$$

and so

(3-11)
$$\sum_{n=0}^{\infty}\frac{1}{f_{p2^n}}=\frac{1}{f_p}+\frac{\sqrt{5}}{2}\frac{\phi^{-p}}{\sinh(p\psi)}=\begin{cases}\frac{1}{f_p}+\frac{\sqrt{5}}{l_p}\phi^{-p}, & p \text{ odd}\\ \frac{1}{f_p}+\frac{1}{f_p}\phi^{-p}, & p \text{ even}\end{cases}$$

With respect to $\phi^{-p}=(-1)^p\frac{1}{2}\left(l_p-\sqrt{5}f_p\right)$ we further obtain

(3-12)
$$\sum_{n=0}^{\infty}\frac{1}{f_{p2^n}}=\begin{cases}\frac{1}{2}\left(\frac{2}{f_p}+5\frac{f_p}{l_p}-\sqrt{5}\cdot\right), & p \text{ odd}\\ \frac{1}{2}\left(\frac{2+l_p}{f_p}-\sqrt{5}\right), & p \text{ even}\end{cases}$$

where the term $\frac{2}{f_p}+5\frac{f_p}{l_p}$ may be replaced by $\frac{2l_p+5f_p^2}{f_p l_p}=\frac{2l_p+5l_p^2+4}{f_p l_p}=\frac{(l_p+1)^2+3}{f_{2p}}$. □

The compact formula (3-3) follows from (3-11) by replacing the sinh

$$\frac{1}{f_p}+\frac{\sqrt{5}}{2}\frac{\phi^{-p}}{\sinh(p\psi)}=\frac{1}{f_p}+\frac{\sqrt{5}}{\phi^{2p}-1}.$$

Finally, we will prove two other statements which also concerns Fibonacci sums with power indices.

Theorem 3-2

For integer p>1 the following two statements holds true:

(i)
$$\sum_{n=0}^{\infty}\frac{f_{p^{n+1}-p^n}}{f_{p^{n+1}}f_{p^n}}=\frac{1}{f_p}\left(f_{p-1}+\varphi^{-p}\right)$$
$$=\varphi^{-1}+\left(1+(-1)^p\right)\frac{\varphi^{-p}}{f_p}$$

(ii)
$$\sum_{n=0}^{\infty}\frac{f_{p^{n+1}-p^n}}{l_{p^{n+1}}l_{p^n}}=\frac{1}{l_p}\left(f_{p-1}+\frac{\varphi^{-p}}{\sqrt{5}}\right)$$
$$=\frac{\varphi^{-1}}{\sqrt{5}}+\left(1+(-1)^p\right)\frac{\varphi^{-p}}{l_p\sqrt{5}}$$

For p=2 the series (i) is identical with the Millin series provided summation begins with n=1 instead of n=0 there, because it is $\frac{f_{2^{n+1}-2^n}}{f_{2^{n+1}}f_{2^n}}=\frac{f_{2^n}}{f_{2^{n+1}}f_{2^n}}=\frac{1}{f_{2^{n+1}}}$.

A special case of interest is p=3. Then we get $\frac{f_{3^{n+1}-3^n}}{f_{3^{n+1}}f_{3^n}}=\frac{f_{2\cdot 3^n}}{f_{3^{n+1}}f_{3^n}}=\frac{f_{3^n}l_{3^n}}{f_{3^{n+1}}f_{3^n}}=\frac{l_{3^n}}{f_{3^{n+1}}}$ and so the series (i) becomes to $\sum_{n=0}^{\infty}\frac{l_{3^n}}{f_{3^{n+1}}}=\varphi^{-1}$.

An analogous statement holds true for the series (ii) which varies to $\sum_{n=0}^{\infty}\frac{f_{3^n}}{l_{3^{n+1}}}=\frac{\varphi^{-1}}{\sqrt{5}}$ for p=3. This follows immediately from $\frac{f_{3^{n+1}-3^n}}{l_{3^{n+1}}l_{3^n}}=\frac{f_{2\cdot 3^n}}{l_{3^{n+1}}l_{3^n}}=\frac{f_{3^n}l_{3^n}}{l_{3^{n+1}}l_{3^n}}=\frac{f_{3^n}}{l_{3^{n+1}}}$.

It is remarkable that for odd parameters p the series (i) always sums up to the reciprocal of the golden ratio ϕ^{-1}, independent from p. Similarly noteworthy: under the same circumstances the sum of the series (ii) always equals $\frac{\phi^{-1}}{\sqrt{5}}$.

The proof of Theorem 3-2 is based on the well known Fibonacci-Lucas subtraction formula

(3-13) $$f_m l_n - l_m f_n = 2\cdot(-1)^n f_{m-n}.$$

To show the power of the representation introduced in section 1 we will prove this formula here in terms of sine and cosine hyberbolic. Certainly, formula (3-13) follows easily from the theorems of addition and subtraction for sinh and cosh. The appropriate formulas are listed below for the sake of completeness.

(3-14) $$\sinh(m\psi)\cosh(n\psi)-\cosh(m\psi)\sinh(n\psi)=\sinh((m-n)\psi)$$
(3-15) $$\cosh(m\psi)\cosh(n\psi)-\sinh(m\psi)\sinh(n\psi)=\cosh((m-n)\psi)$$

Therefore, with respect to the sinh-cosh-representation we get

(3-16)

$$\frac{\sqrt{5}}{2} f_m \cdot \frac{1}{2} l_n - \frac{1}{2} l_m \cdot \frac{\sqrt{5}}{2} f_n = \frac{\sqrt{5}}{2} f_{m-n} \qquad \text{for m, n even, by (3-14)}$$
$$\frac{\sqrt{5}}{2} f_m \cdot \frac{1}{2} l_n - \frac{1}{2} l_m \cdot \frac{\sqrt{5}}{2} f_n = -\frac{\sqrt{5}}{2} f_{m-n} \qquad \text{for m, n odd, by (3-14)}$$
$$\frac{\sqrt{5}}{2} f_m \cdot \frac{1}{2} l_n - \frac{1}{2} l_m \cdot \frac{\sqrt{5}}{2} f_n = \frac{\sqrt{5}}{2} f_{m-n} \qquad \text{for m odd, n even, by (3-15)}$$
$$\frac{\sqrt{5}}{2} f_m \cdot \frac{1}{2} l_n - \frac{1}{2} l_m \cdot \frac{\sqrt{5}}{2} f_n = -\frac{\sqrt{5}}{2} f_{m-n} \qquad \text{for m even, n odd, by (3-15)}$$

Dividing both sides of the relations (3-16) by $\sqrt{5}$ and multiplying by 2 results in formula (3-13) immediately.

<u>Proof of Theorem 3-2:</u> Replacing *m* and *n* in formula (3-13) by p^{n+1} and p^n respectively gives

(3-17)
$$f_{p^{n+1}} l_{p^n} - l_{p^{n+1}} f_{p^n} = 2 \cdot (-1)^{p^n} f_{p^{n+1}-p^n}$$

which implies

(3-18)
$$\frac{f_{p^{n+1}-p^n}}{f_{p^{n+1}} f_{p^n}} = \frac{(-1)^p}{2} \left(\frac{l_{p^n}}{f_{p^n}} - \frac{l_{p^{n+1}}}{f_{p^{n+1}}} \right), \text{ for n>0}$$

For the partial sum of the series (i) then we obtain

(3-19)
$$\begin{aligned}
\sum_{n=0}^{N} \frac{f_{p^{n+1}-p^n}}{f_{p^{n+1}} f_{p^n}} &= \frac{f_{p-1}}{f_p} + \frac{(-1)^p}{2} \sum_{n=1}^{N} \left(\frac{l_{p^n}}{f_{p^n}} - \frac{l_{p^{n+1}}}{f_{p^{n+1}}} \right) \\
&= \frac{f_{p-1}}{f_p} + \frac{(-1)^p}{2} \left(\frac{l_p}{f_p} - \frac{l_{p^{N+1}}}{f_{p^{N+1}}} \right) \\
&= \frac{f_{p-1}}{f_p} + \frac{f_{p^{N+1}-p}}{f_{p^{N+1}} f_p} \\
&= \frac{1}{f_p} \left(f_{p-1} + \frac{f_{p^{N+1}-p}}{f_{p^{N+1}}} \right)
\end{aligned}$$

For $N \to \infty$ the term $\frac{f_{p^{N+1}-p}}{f_{p^{N+1}}}$ tends to ϕ^{-p}, which implies

(3-20)
$$\sum_{n=0}^{\infty} \frac{f_{p^{n+1}-p^n}}{f_{p^{n+1}} f_{p^n}} = \frac{1}{f_p} \left(f_{p-1} + \varphi^{-p} \right)$$

From this we get the equality with $\varphi^{-1} + \left(1 + (-1)^p\right) \frac{\varphi^{-p}}{f_p}$ by replacing $\phi^{-p} = -\frac{1}{2}\left(l_p - \sqrt{5} f_p\right)$ for odd *p* and replacing $\phi^{-p} = 2\phi^{-p} - \frac{1}{2}\left(l_p - \sqrt{5} f_p\right)$ for even *p*. Thus, assertion (i) has been proved.

The proof of the sum (ii) may be accomplished very analogue. From formula (3-17) now we get

(3-21) $$\frac{f_{p^{n+1}-p^n}}{l_{p^{n+1}}l_{p^n}} = \frac{(-1)^p}{2}\left(\frac{f_{p^n}}{l_{p^n}} - \frac{f_{p^{n+1}}}{l_{p^{n+1}}}\right), \text{ for n>0}$$

Thus we obtain

(3-22) $$\begin{aligned}\sum_{n=0}^{N}\frac{f_{p^{n+1}-p^n}}{l_{p^{n+1}}l_{p^n}} &= \frac{f_{p-1}}{l_p} + \frac{(-1)^p}{2}\sum_{n=1}^{N}\left(\frac{f_{p^n}}{l_{p^n}} - \frac{f_{p^{n+1}}}{l_{p^{n+1}}}\right)\\ &= \frac{f_{p-1}}{l_p} + \frac{(-1)^p}{2}\left(\frac{f_p}{l_p} - \frac{f_{p^{N+1}}}{l_{p^{N+1}}}\right)\\ &= \frac{f_{p-1}}{l_p} + \frac{f_{p^{N+1}-p}}{l_{p^{N+1}}l_p}\\ &= \frac{1}{l_p}\left(f_{p-1} + \frac{f_{p^{N+1}-p}}{l_{p^{N+1}}}\right)\end{aligned}$$

for the partial sum of the series (ii).
For $N \to \infty$ the term $\frac{f_{p^{N+1}-p}}{l_{p^{N+1}}}$ tends to $\frac{\phi^{-p}}{\sqrt{5}}$, and therefore

(3-23) $$\sum_{n=0}^{\infty}\frac{f_{p^{n+1}-p^n}}{l_{p^{n+1}}l_{p^n}} = \frac{1}{l_p}\left(f_{p-1} + \frac{\varphi^{-p}}{\sqrt{5}}\right)$$

Similar to the argumentation above we get the equality with $\frac{\varphi^{-1}}{\sqrt{5}} + \left(1+(-1)^p\right)\frac{\varphi^{-p}}{\sqrt{5}f_p}$ by replacing again $\phi^{-p} = -\frac{1}{2}\left(l_p - \sqrt{5}f_p\right)$ for odd *p* and replacing $\phi^{-p} = 2\phi^{-p} - \frac{1}{2}\left(l_p - \sqrt{5}f_p\right)$ for even *p*. This proves the assertion (ii) of Theorem 3-2. ☐

Without any reference to the golden ratio the sums (i) and (ii) of Theorem 3-2 can be expressed in perfect symmetry by

(3-24) $$\sum_{n=0}^{\infty}\frac{f_{p^{n+1}-p^n}}{f_{p^{n+1}}f_{p^n}} = \begin{cases}\frac{1}{2}\left(2\frac{l_p}{f_p} - 1 - \sqrt{5}\right), & \text{if } p \text{ even}\\ \frac{1}{2}\left(\sqrt{5}-1\right), & \text{if } p \text{ odd}\end{cases}$$

and

(3-25) $$\sum_{n=0}^{\infty}\frac{f_{p^{n+1}-p^n}}{l_{p^{n+1}}l_{p^n}} = \begin{cases}\frac{1}{2}\left(\frac{1}{\sqrt{5}} + 1 - 2\frac{f_p}{l_p}\right), & \text{if } p \text{ even}\\ \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right), & \text{if } p \text{ odd}\end{cases}$$

Both relations yield from the formulas (i) and (ii) of Theorem 3-2 by replacing $\phi^{-p} = \frac{1}{2}\left(l_p - \sqrt{5}f_p\right)$ for the case of even *p*. For odd *p* all is apparently true anyway.